February 2, 2008

Constraining New Forces in the Casimir Regime Using the Isoelectronic Technique

R. S. Decca,[1,*] D. López,[2] H. B. Chan, [3] E. Fischbach,[4] D. E. Krause,[5,4] and C. R. Jamell[1]

[1]*Department of Physics, Indiana University-Purdue University Indianapolis, Indianapolis, Indiana 46202, USA*

[2]*Bell Laboratories, Lucent Technologies, Murray Hill, New Jersey 07974, USA*

[3]*Department of Physics, University of Florida, Gainesville, Florida 32611, USA*

[4]*Department of Physics, Purdue University, West Lafayette, Indiana 47907, USA*

[5]*Physics Department, Wabash College, Crawfordsville, Indiana 47933, USA*

Abstract

We report the first isoelectronic differential force measurements between a Au-coated probe and two Au-coated films, made out of Au and Ge. These measurements, performed at submicron separations using soft microelectromechanical torsional oscillators, eliminate the need for a detailed understanding of the probe-film Casimir interaction. The observed differential signal is directly converted into limits on the parameters α and λ which characterize Yukawa-like deviations from Newtonian gravity. We find $\alpha \lesssim 10^{12}$ for $\lambda \sim 200$ nm, an improvement of ~ 10 over previous limits.

Although gravity was the first fundamental force to be understood, the quest to unify it with the other fundamental forces has remained elusive. One of the reasons is the apparent weakness of the gravitational interaction at small separations. Consequently, a significant number of experimental searches for new forces over ultra-short distances has been performed [1, 2, 3, 4, 5, 6, 7, 8, 9]. They have been stimulated by at least three different—but related— motivations:

(a) Some unification theories, incorporating n compact extra spatial dimensions with characteristic size $R(n)$, predict deviations from Newtonian gravity over sub-mm scales [10, 11, 12]. Some extra-dimensional theories characterize the deviations by a Yukawa-modified potential $V(r)$ [10, 11, 12],

$$V(r) = V_N(r)[1 + \alpha e^{-r/\lambda}] , \tag{1}$$

where $V_N(r)$ is the Newtonian gravitational potential for two point masses separated by a distance $r \gg R(n)$, and α and $\lambda \sim R(n)$ are constants. Thus, the extra-dimensional theories of Ref. [10] provide a parameter $R(n)$, which for $n > 1$ is naturally small, and an associated constant α_n which is relatively poorly constrained.

(b) String theory and other extensions to the Standard Model predict the existence of new light bosons such as dilatons, moduli, radions, cosmons, and bulk gauge bosons [11]. The exchange of these massive particles leads to corrections to gravity as in Eq. (1), with $\alpha \gg 1$ and λ, related to the boson mass m by $\lambda = \hbar/mc$, as large as a few microns. Hence, the limits on α may provide useful guidance in narrowing down the myriad possible models linking physics at very high energy scales to the much lower energy Standard Model.

(c) Theories in which these hypothetical new forces arise from an inverse-power potential, $V \propto r^{-p}$. An extensive review of such inverse-power-law forces has been given recently in Ref. [13].

Existing limits on such forces, when parameterized as in Eq. (1), are relatively weak for several reasons. Most significantly, if λ is small then the effective interacting masses are themselves necessarily small, and background disturbances play a relatively more important role. Secondly, for experiments with typical separations $\lesssim 1\mu$m the dominant background arises from the Casimir force [4, 14], which is not only relatively strong over the relevant distances, but is also somewhat difficult to completely characterize at the required level of precision [1]. Absent any alternative, limits on $\alpha = \alpha(\lambda)$ have been inferred by developing

detailed theoretical models of the Casimir interaction [1], yielding the limit $\alpha \overset{<}{\sim} 10^{13}$ for $\lambda \cong 150$ nm. Further improvements in experiments and in our understanding of the Casimir force will lead to stronger constraints on $\alpha(\lambda)$, but it is better to sidestep theory altogether by carrying out what amounts to a "Casimir-less" measurement.

In this Letter we report isoelectronic measurements (IET) [3, 15], where the Casimir background is subtracted at the outset, thus avoiding the necessity to model the Casimir force. IET exploit the essentially electronic nature of the Casimir force, whereas gravity and hypothetical forces couple to nucleons and electrons. Hence, vacuum fluctuations cannot account for any significant difference in the forces between a probe and two materials with identical electronic properties.

A schematic of our set-up is shown in Fig. 1. We compare the force *differences* over two dissimilar materials, Au and Ge, which have been coated with a common layer of Au of thickness $d_{Au}^p = 150$nm $> \lambda_p$, where $\lambda_p = 135$ nm is the plasma wavelength for Au. The fractional difference of the Casimir force between two infinitely thick metallic plates and two plates of thickness d_{Au}^p is $\sim e^{-4\pi d_{Au}^P/\lambda_p} \sim 10^{-6}$ [16]. In our experiment, it translates to a difference $\Delta F_C \overset{<}{\sim} 10^{-17}$ N in the Casimir force between the Au coated sphere and the two sides of the Au/Ge composite sample. Hence any differential signal that the probe detects as it oscillates over the underlying Au and Ge (which provide a large mass density difference, $\rho_{Au} - \rho_{Ge} = 13.96 \times 10^3 \, \text{kg/m}^3$) must be due to an interaction via either gravity or some new hypothetical force. Thus by directly comparing the forces on the Au and Ge substrates a limit on $\alpha(\lambda)$ can be obtained, without having to resort to a theory of the Casimir force for real materials. The expression for this hypothetical force difference is

$$
\begin{aligned}
\Delta F^{hyp}(z) &= -4\pi^2 G\alpha\lambda^3 e^{-z/\lambda} RK_s K_p, \\
K_s &= \left[\rho_{Au} - (\rho_{Au} - \rho_{Cr})\, e^{-d_{Au}^s/\lambda} - (\rho_{Cr} - \rho_s)\, e^{-(d_{Au}^s + d_{Cr})/\lambda}\right], \\
K_p &= \left[(\rho_{Au} - \rho_{Ge})\, e^{-\left(d_{Au}^p + d_{Pt}\right)/\lambda} \left(1 - e^{-d_{Ge}/\lambda}\right)\right],
\end{aligned}
\tag{2}
$$

where G is the gravitational constant, $R \sim 50\mu$m is the radius of the sphere, K_s (K_p) is a term associated only with the layered structure of the sphere (plate), ρ_s, ρ_{Cr}, ρ_{Au}, and ρ_{Ge} are the densities of the sapphire sphere, a Cr layer (used to increase Au adhesion to sapphire), the Au layers and the Ge layer, respectively. Thicknesses d_i for the different materials are given in Fig. 1.

Since the hypothetical forces under study are weak [1], a high sensitivity force measurement is required. A microelectromechanical torsional oscillator (MTO) with a soft spring κ (high force resolution) and high quality factor (Q) satisfies the required demands. The MTO has a low coupling with the environment [1, 2] $\kappa \sim 10^{-9}\,\mathrm{Nm/rad}$, and $Q \sim 10^4$.

The films deposited on the sphere and the MTO were characterized by atomic force microscopy (AFM). A typical AFM line-cut at the interface between the Au and Ge layers is shown in Fig. 1b. The observed ridge (a valley in some samples) arises from the imperfect alignment of the mask when depositing the Au and Ge. The analysis of the AFM images indicates the granular character of the samples, showing a maximum height difference of $22\,\mathrm{nm}$. The average lateral dimension of the grains was in the 100-150 nm range, although grains as large as $500\,\mathrm{nm}$ were observed. The uncertainty in the position of the zero height level (with respect to which the roughness is measured [1]) is $\sim 0.2\,\mathrm{nm}$.

The experimental arrangement and calibrations performed are very similar to those previously used to determine Casimir forces [1, 2]. A voltage was applied to the sphere to eliminate the residual electrostatic force caused by the difference in work functions between the Au layer on the MTO and the sphere. The angular displacement of the MTO, determined by measuring the difference in capacitance between the MTO and the underlying electrodes, $\Delta C = C_{right} - C_{left}$, yielded the force acting on it. The sensitivity in the angular deviation is $\delta\theta \simeq 10^{-9}\mathrm{rad}/\sqrt{\mathrm{Hz}}$.

The force sensitivity is improved if the measurements are performed at resonance, $\omega = \omega_o$. In this case, the minimum detectable force is dominated by thermal fluctuations, $\delta F(\omega_o) \simeq \delta F_{thermal} = 1/b\sqrt{4\kappa k_B T/(\omega_o Q)}$, where k_B is Boltzmann's constant. Consequently, it is necessary to measure the effect of ΔF_{hyp} at resonance. ΔF_{hyp} can be described as the product of a function of the separation $e^{-z/\lambda}$ and a function of the $x-$coordinate, the difference in the average mass density. Hence, we induced a vertical oscillation on the MTO such that the separation between the MTO and the sphere changed as $z_m = z_{mo} + \delta z \cos(\omega_z t)$, with $z_{mo} \gg \delta z$. We simultaneously moved the MTO along a direction parallel to its axis, such that the effective mass density under the sphere was $\rho_{eff} = \rho^+ + \rho^- \Xi(t)$, where $\rho^\pm = (\rho_{Au} \pm \rho_{Ge})/2$, and $\Xi(t)$ is a square-wave function with characteristic angular period $T_x = 2\pi\omega_x^{-1}$. At $t = 0$ the sphere is positioned over the MTO on the Au/Au half. The Casimir interaction leads to a shift of the resonance frequency of the MTO from its natural oscillation frequency f_o

to f_r [17]. By selecting $\omega_z + \omega_x = \omega_r$, ΔF_{hyp} has a Fourier component at ω_r given by

$$\mathcal{F}_{hyp}(z_{mo}, \omega_r) = \Delta F_{hyp}(z_{mo}) \times \frac{2}{\pi} \times I_1\left(\frac{\delta z}{\lambda}\right), \tag{3}$$

where I_1 is a Bessel function of the second kind. $\mathcal{F}_{hyp}(\omega_r)$ is the only Fourier component with a significant signal-to-noise ratio, even though no parts in the system are moving at f_r. We selected $f_x = \omega_x/2\pi = \omega_r/140\pi = f_r/70 \sim 10\,\text{Hz}$. Consequently, $f_z = \omega_z/2\pi = 69 f_x$. The phases of the different signals were chosen to simultaneously cross through 0 every $t_{xr} \equiv 70/f_r$ ($t_{xz} \equiv 69/f_z$) for the signals at f_x and f_r (f_z). The synthesized signal at f_r was used as reference, see Fig. 2a. The amplitudes were adjusted to provide a peak-to-peak lateral displacement of $D \simeq 150\mu\text{m}$ and a vertical amplitude that ranged between $\delta z \simeq 10\,\text{nm}$ at the closest separations to $\delta z \simeq 50\,\text{nm}$ at $z_{mo} = 500\,\text{nm}$.

For each resonant period of oscillation of the MTO, $T_r = 1/f_r$, ten equally spaced data points $\mathcal{F}_d(t_i)$ were acquired, $t_i = iT_r/10$, $i = 1, \cdots, 10$. Simultaneously, $\mathcal{P}(t_i) = \mathcal{F}_d(t_i)\cos\omega_r t_i$ and $\mathcal{Q}(t_i) = \mathcal{F}_d(t_i)\sin\omega_r t_i$ were determined. Averaging of $\mathcal{F}_d$, $\mathcal{P}$, and $\mathcal{Q}$ was achieved by adding the signals for all different T_r in the corresponding $i = 1, \cdots, 10$ intervals. Furthermore, the summation over t_i of $\mathcal{P}(t_i)$ ($\mathcal{Q}(t_i)$) yields the in-phase $\mathcal{F}_p$ (quadrature $\mathcal{F}_q$) Fourier component at ω_r.

A strong reduction of the random noise was achieved by increasing the integration times τ. τ was changed between 0.1 and 2000 s in a 1, 2, 5, 10, $\cdots$ sequence. Fig. 2 shows the results obtained at a separation $z_{mo} = 300$ nm. Figs. 2b-d show the observed behavior of $\mathcal{F}_p$ and $\mathcal{F}_q$ in one sample for three characteristic values of τ (the number of repetitions N decreased as τ increased). Fig. 2d, shows the behavior at $\tau = 2,000\,\text{s}$ for all seven samples investigated. Several features should be noted in Fig. 2: (i) for each sample and all values of τ $|\overline{\mathcal{F}}_T| = \sqrt{\overline{\mathcal{F}}_p^2 + \overline{\mathcal{F}}_q^2}$ is constant within the statistical error. $\overline{\mathcal{F}}_p$ ($\overline{\mathcal{F}}_q$) is the average of $\mathcal{F}_p$ ($\mathcal{F}_q$) over the different repetitions; (ii) $|\overline{\mathcal{F}}_T|$ remains constant within a factor of 2 for different samples; (iii) the phase $\Theta = \arctan(\overline{\mathcal{F}}_q/\overline{\mathcal{F}}_p)$, however, assumes values close to either 0 or π, indicating that the force is larger over either the Au or the Ge side of the composite sample, respectively. We assume that the forces measured correspond to either $\Theta = 0$ or π [18].

We calculated the force $\overline{\mathcal{F}}(\tau, z_{mo}) = \overline{\mathcal{F}}_T(\tau, z_{mo}) \pm s_{N^*}(\tau, z_{mo})t_{\beta,N^*}$ required to exclude, at the $\beta = 95\,\%$ confidence level, any hypothetical force of the form given by Eq. (2). Here $s_{N^*}(\tau, z_{mo})$ is the mean square error of $\overline{\mathcal{F}}_T(\tau, z_{mo})$, t_{β,N^*} is the Student's coefficient, and $N^* = 6$ is the number of repetitions at $\tau = 2000\,\text{s}$. The + (-) sign is is used for the in

phase (π out of phase) cases. Fig. 3 shows the asymptotic behavior $\overline{\mathcal{F}}(\tau = 2,000\,\text{s}, z_{mo})$ for all samples. $|\overline{\mathcal{F}}(\tau = 2,000\,\text{s}, z_{mo})|$ decreases as z_{mo} increases, but not exponentially, as expected from Eq. (3). Independently of the origin of $\overline{\mathcal{F}}(\tau = 2,000\,\text{s}, z_{mo})$, however, the shaded region in Fig. 3 represents values of hypothetical forces *excluded* by our experiment.

Although $\overline{\mathcal{F}}(\tau = 2,000\,\text{s}, z_{mo}) \neq 0$, we do not believe it originates from new physics. As noted above, its z_{mo} dependence is not exponential. More importantly, it changes sign depending on the sample under study, showing no correlation with the underlying Au and Ge layers. We therefore attribute its presence to the manifestation in the Casimir force of a change $\delta z' \sim 0.1\,\text{nm}$ in z_{mo}, i.e. z_{mo} takes different values over the Au and Ge sides. With the AFM characterization, the height difference between the two sides is not known to better than $\delta z' \sim 0.1\,\text{nm}$ [19]. This translates into a residual Casimir force difference $|\Delta F_C(200\text{nm})| \sim 15$ fN, sufficient to explain the observed $|\overline{\mathcal{F}}(\tau, z_{mo})|$.

We can, however, rule out other sources for the observed background: *(i)* A motion of the sphere in a direction not parallel to the MTO's axis was ruled out by performing the experiments without crossing the interface, and at different z_{mo}. The uncertainty $\delta y \sim 3\,\text{nm}$ in the motion of the sphere over the $D = 150\,\text{nm}$ excursion yields a $\sim 0.2\,\text{fN}$ error. *(ii)* Local differences in roughness yield an estimate for the residual force to be $\lesssim 2\,\text{fN}$, while patch potentials [21] give a contribution $\lesssim 5\,\text{fN}$. The effect of roughness and patch potentials can be further reduced by moving the sphere back and forth across the interface to points *randomly* selected, with dispersion large enough to average the local changes. This approach yielded the same results. *(iii)* The effect of the ridge at the interface is also small. A Fourier analysis of its contribution yields $\Phi(200\,\text{nm}) \lesssim 3\,\text{fN}$. As z_{mo} increases, this contribution decreases rapidly, having a value $\Phi(500\,\text{nm}) \lesssim 0.1$ fN. We note that for samples with a valley instead of a ridge the contribution is negligible. *(iv)* We also checked that the finite size of the sample does not affect the Casimir background or the Yukawa corrections. *(v)* Magnetic or gravitational (Newtonian) forces do not give a measurable background at ω_r. The component of the magnetic force with a signal at ω_r is associated with the preferential presence of magnetic impurities in one of the sides of the engineered sample (either Ge or Au). This force, comparable to the magnetic interaction between isolated atoms, is much smaller than the sensitivity of our apparatus. The Newtonian gravitational attraction difference between the sphere and the composite Au/Ge sample is $\sim 3 \times 10^{-21}$ N, independent of separation. Hence, this force not only is too small to be detected, but it also

does not provide a Fourier component at ω_r.

The results shown in Fig. 3 can be used to obtain more stringent limits on hypothetical forces since these must be less than, or at most equal to, the observed background. The values of forces in the shaded region in Fig. 3 have been experimentally excluded. Furthermore, the $\Theta = 0$ and $\Theta = \pi$ cases yield very similar absolute values for $\overline{\mathcal{F}}$. Hence, for a given z_{mo}, we use the smallest $|\overline{\mathcal{F}}(\tau = 2,000\,\text{s}, z_{mo})|$ as the maximum allowed hypothetical force. Using this value in the left-hand side of Eq. (3) we obtain an $\alpha(\lambda)$ curve. Repeating this procedure for different z_{mo} we obtain a family of curves whose envelope provides the strictest limits arising from our experiments. This curve, together with previous results, is shown in Fig. 4, which also shows the regions in λ, α phase space where different models predict the existence of new forces [22]. Our realization of a "Casimir-less" IET yields a ~ 10-fold improvement (in the [30, 400] nm range) on existing limits for Yukawa-like corrections to Newtonian gravity. This has significant consequences for models of moduli exchange, as proposed by supersymmetry, by further constraining the supersymetric parameters. We believe that our direct, improved experimental test at submicron separations will continue to motivate theoretical development for this range of separations. We also note that our experiment can be improved by gluing the sphere to the MTO and oscillating the test masses over it. By judiciously designing the test masses, most of the problems associated with the background can be removed. In this scenario, limits on α down to 10^6 can be achieved at separations $z_{mo} \sim 100\,\text{nm}$.

* Electronic address: rdecca@iupui.edu

[1] R. S. Decca *et al.*, Phys. Rev. D **68**, 116003 (2003), and references therein.

[2] R. S. Decca *et al.*, Phys. Rev. Lett. **91**, 050402 (2003).

[3] E. Fischbach *et al.*, Phys. Lett. A 318, 165 (2003).

[4] M. Bordag, U. Mohideen, and V.M. Mostepanenko, Phys. Rep. **353**, 1 (2001), and references therein.

[5] J.C. Long, H.W. Chan, and J.C. Price, Nucl. Phys. **B539**, 23 (1999).

[6] S.K. Lamoureaux, Phys. Rev. Lett. **78**, 5 (1997); **81**, 5475(E) (1998)

[7] U. Mohideen and A. Roy, Phys. Rev. Lett. **81**, 4549 (1998); G.L. Klimchitskaya *et al.*, Phys. Rev. **A60**, 3487 (1999); A. Roy and U. Mohideen, Phys. Rev. Lett. **82** 4380 (1999); B.W. Harris, F. Chen, and U. Mohideen, Phys. Rev. **A62**, 052109 (2000).

[8] J. Chiaverini *et al.*, Phys. Rev. Lett. **90**, 151101 (2003).

[9] G. Bressi *et al.*, Phys. Rev. Lett. **88**, 041804 (2002).

[10] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. **B429**, 263 (1998); Phys. Rev. **59**, 086004 (1999); I. Antoniandis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Rev. **B436**, 257 (1998).

[11] E. G. Adelberger, B. R. Heckel, and A. E. Nelson, Ann. Rev. Nucl. Part. Sci., **53**, 77 (2003).

[12] B. Abbott *et al.*, Phys. Rev. Lett. **86**, 1156 (2001).

[13] D.E. Krause and E. Fischbach in *Particle Physics and the Universe*, Ed. by J. Trampetić and J. Wess (Springer, Berlin, 2004).

[14] H.B.G. Casimir, Proc. K. Ned. Akad. Wet. **B51**, 793 (1948).

[15] E. Fischbach *et al.*, Class. Quantum Grav. **18**, 2427 (2001); D.E. Krause and E. Fischbach, Phys. Rev. Lett. **89**, 190406 (2002).

[16] R. Matloob and H. Falinejad, Phys. Rev. A **64**, 042102 (2001). The differences between the model used in this paper and more sophisticated calculations of the Casimir force agree to within 2 %, adequate for our order of magnitude estimate.

[17] f_r is a function of the separation due to the non-linear dependence of the Casimir force [1, 2].

[18] The deviations of Θ with respect to 0 and π are comparable to those observed in the system when the experiment is replaced by a frequency mixer. This leads us to believe that these deviations are artificial and induced by the electronic circuitry.

[19] $\delta z'$ is not expected to be originated in curvature differences between the MTO and the platform. These differences, along the direction of motion of the sphere, are smaller than $0.1\,$nm, the measurement error (WYCO NT3300 optical profilometer). The sample dependant curvatures are a few nm over the $500\,\mu$m of the MTO's length.

[20] V. B. Bezerra, G. L. Klimchitskaya, and C. Romero, Mod. Phys. Lett. **A12**, 2613 (1997).

[21] F. Chen *et al.*, Phys. Rev. A **69**, 022117 (2004).

[22] S. Dimopoulos and A. A. Geraci, Phys. Rev. D **68**, 124021 (2003). Limits on moduli shown in Fig. 4 were obtained by removing the constraints imposed on the undetermined constants $(\lambda^{(0)}/\lambda^{(1)}) \times (M/5 \times 10^{17}\,\text{GeV})$ and $(\lambda_g^{-1}) \times (M/5 \times 10^{17}\,\text{GeV})$. While it is unlikely that they extend beyond the range 10^{-2}–10^2 imposed by the authors, experiments should constrain the values of undetermined parameters in a theory.

FIG. 1: **(a)** Scanning electron microscope image of the MTO with the composite sample deposited on it. Inset: Schematic of the sample deposited on the MTO. The coordinate system used in the paper is indicated. **(b)** Diagram of the experimental set-up. Inset: AFM profile of the sample across the interface. $d_{Cr} = 1\,\text{nm}$; $d_{Au}^s = 200\,\text{nm}$: Au layer thickness on the sapphire probe; $d_{Ge} = 200\,\text{nm}$: thickness of both the Ge (indicated in orange) and bottom Au layers on the substrate; $d_{Pt} = 1\,\text{nm}$: thickness of the Pt film (used to avoid diffusion of Ge in Au). There is also a $d_{Ti} = 1\,\text{nm}$ layer of Ti deposited to increase adhesion to the MTO.

FIG. 2: **(a)** Integration time, τ, dependence of the signal. Data at the shortest integration time have been scaled by $A = 20$. Data were obtained at $z_{mo} = 300\,\text{nm}$. The dotted line is the measured reference signal at ω_r, adjusted to fit on the same scale. **(b)** through **(d)** Measured $\mathcal{F}_p$ and $\mathcal{F}_q$ signals with $\tau = 2\,\text{s}$ ($N = 300$), $200\,\text{s}$ ($N = 30$), and $2,000\,\text{s}$ ($N = 6$), respectively. Each point is a repetition of the experiment. The circles in **(b)** and **(c)** are centered at $(\overline{\mathcal{F}}_p, \overline{\mathcal{F}}_q)$ and have a radius given by the calculated thermal noise over the relevant τ. The measured noise, obtained at large z_{mo}, exceeds the thermal noise by $\sim 20\,\%$. **(d)** Results at $\tau = 2000\,\text{s}$ for all samples.

FIG. 4: Values in the λ, α phase space excluded by the experiment. The red curve represents the limit obtained in the current experiment. Curves 1 to 5 were obtained by Mohideen's group [7], our group [1], Lamoreaux [6], Kapitulnik's group [8], and Price's group [5], respectively.

FIG. 3: Dependence of the peak-to-peak Fourier component $\overline{\mathcal{F}}_T(\tau = 2,000\,\text{s}, z_{mo})$ on z_{mo} for all measured samples. The shaded region in the figure represents values of hypothetical forces excluded by our experiments. Colors and symbols for the different samples are the same as those used in Fig. 2d. The upper (lower) hatched area is defined by the points with minimum absolute value of $\overline{\mathcal{F}}(\tau, z_{mo}) = \overline{\mathcal{F}}_T(\tau, z_{mo}) + s_{N^*}(\tau, z_{mo})t_{\beta,N^*}$ $(\overline{\mathcal{F}}(\tau, z_{mo}) = \overline{\mathcal{F}}_T(\tau, z_{mo}) - s_{N^*}(\tau, z_{mo})t_{\beta,N^*})$.